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Ground State Energy of Dilute Bose Gas in Small Negative Potential Case

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Abstract It is well known that the ground state energy of a three dimensional dilute Bose gas in the thermodynamic limit is $E = 4\pi a\rho N$ when the particles interact via a non-negative, finite range, spherically symmetric, two-body potential. Here, N is the number of particles, ρ is the density of the gas, and a is the scattering length of the potential. In this paper, we prove the same result without the non-negativity condition on the potential, provided the negative part is small.

Keywords Ground state energy · Dilute Bose gas · Negative potential

1 Introduction

The ground state of a dilute Bose gas yields many interesting results. Dyson's estimate [2] of the ground state energy was the first rigorous one, and about 40 years later, Lieb and Yngvason [4] proved the correct leading term for the lower bound. Similar results for two-dimensional case were also proved subsequently [5].

All these works, however, were done under the assumption of non-negative interaction potential. In this paper, we relax this condition by permitting a small negative component in the potential. To see this more precisely, we introduce the following model.

We consider a system of N interacting three-dimensional Bosons in a three dimensional torus $\mathbb{T}$ of side length L. Given the two-particle interaction V, the Hamiltonian of this system is

$$H = -\sum_{j=1}^{N} \Delta_j + \sum_{i<j}^{N} V(x_i - x_j) \tag{1.1}$$

where $x_i \in \mathbb{T}$ are the positions of the particles and Δ_i denotes the Laplacian with respect to i-th particle. Every Bosonic state function in this paper is symmetric, smooth, and L^2-normalized. The potential V is spherically symmetric, continuous, and compactly supported.

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The ground state energy is defined as follows. For simplicity, we let $X_n = (x_1, \dots, x_n)$ and $dX_n = dx_1 \cdots dx_n$.

Definition 1.1 (The ground state energy) For given Hamiltonian

$$H^{(n)} = -\sum_{j=1}^{n} \Delta_j + \sum_{i<j}^{n} V(x_i - x_j), \tag{1.2}$$

(a) In a bounded set $S \subset \mathbb{R}^3$, its ground state energy, $E(n, S, V)$, with Neumann boundary conditions is defined to be

$$E(n, S, V) = \inf_{\|\psi\|_2=1} \left[\sum_{j=1}^{n} \int_{S^n} dX_n |\nabla_j \psi(X_n)|^2 + \sum_{i<j}^{n} \int_{S^n} dX_n V(x_i - x_j) |\psi(X_n)|^2 \right]. \tag{1.3}$$

(b) In a three-dimensional torus of side length s, $\mathbb{T}_s = \mathbb{R}^3/(s\mathbb{Z})^3$, its ground state energy $E(n, \mathbb{T}_s, V)$ is defined to be

$$E(n, \mathbb{T}_s, V) = \inf_{\|\psi\|_2=1} \left[\sum_{j=1}^{n} \int_{\mathbb{T}_s^n} dX_n |\nabla_j \psi(X_n)|^2 + \sum_{i<j}^{n} \int_{\mathbb{T}_s^n} dX_n V(x_i - x_j) |\psi(X_n)|^2 \right]. \tag{1.4}$$

We want to find the ground state energy of (1.1) in $\mathbb{T} = \mathbb{T}_L$ in the thermodynamic limit, where N and L approach infinity with $\rho = N/L^3$ fixed. When V is non-negative, it is known that the leading term of the ground state energy in Λ, a box of side length L, is $4\pi a \rho N$ as $\rho \to 0$ [7]. To prove it, Λ is divided into small boxes. Each smaller box, Λ_ℓ is of side length ℓ and big enough to have many particles in it. Then, the superadditivity of the ground state energy [4],

$$E(n + n', \Lambda_\ell, V) \geq E(n, \Lambda_\ell, V) + E(n', \Lambda_\ell, V) \tag{1.5}$$

tells that the infimum of the sum of the energies in the small boxes is attained when the particles are evenly distributed among those boxes. The superadditivity (1.5) results from the fact that we can neglect the interactions between n particles and n' particles when finding the lower bound, provided V is non-negative.

Without the non-negativity, however, it is not true in general that adding particles in a box increases the energy. Moreover, it is clear that some negative potential is catastrophic in the sense that no energy lower bound exist. This fact suggests that we need to introduce some other conditions to ensure stability [8].

Definition 1.2 (Stability of potential) A two-particle potential V_0 is stable if there exists $B \geq 0$ such that

$$\sum_{i<j}^{n} V_0(x_i - x_j) \geq -nB \tag{1.6}$$

for all $n \geq 0$ and $x_1, x_2, \dots, x_n \in \mathbb{R}^3$.

Now we can state the main theorem.

Theorem 1.3 *Let V_1 and V_2 be non-negative, spherically symmetric, continuous, compactly supported, two-particle potentials, satisfying*

$$V_1(x) = 0 \quad \text{if } |x| > R_0, \tag{1.7}$$

$$V_2(x) = 0 \quad \text{if } |x| < R_0 \text{ or } |x| > R_1. \tag{1.8}$$

If $V_0 = V_1 - V_2$ is stable, i.e. satisfies (1.6), there exists a small positive constant λ such that, if $V = V_1 - \lambda V_2$ and a is the scattering length of V, then there are positive constants C_0 and ϵ such that the ground state energy of (1.1), with N and L large, has a lower bound

$$E(N, \mathbb{T}, V) \geq 4\pi a\rho N(1 - C_0\rho^{\epsilon}) \tag{1.9}$$

as $\rho \to 0$.

Remark 1.4 From the proof, one can estimate that $\epsilon < \frac{1}{31}$ and $C_0 > 9$ are sufficient for Theorem 1.3. It should also be noted that, however, the error term $C_0\rho^{\epsilon}$ has no significant meaning. λ is independent of ρ and can be estimated from (3.24), (3.36), and (4.1).

Remark 1.5 Corresponding upper bound can be easily obtained with a proper assumption on λ. One can follow the proof of Theorem 2.2 in [7]. For detailed calculation, see [2] and [6]. (Remark 3.1 in [6] explains how to include small negative potential in hard core potential, and one can prove the same result for the potential in Theorem 1.3 using the similar argument in the remark.) More general upper bound calculation for a case of the interaction potential with positive scattering length can be found in [9], which also considers the problem similar to the one in this paper, yet with a different approach.

Remark 1.6 Throughout the paper, C denotes various constants that do not depend on ρ.

2 Outline of the Proof

Step 1: *Non-negativity of the ground state energy in a small box of fixed size. (Lemma 3.3.)*

We first choose the side length of the small box, ℓ, and show that the ground state energy $E(2, S, V_1)$ is bounded below by a function of ℓ as in (3.22) for any rectangular box $S \subset \Lambda_\ell$. To control V_2, the negative potential, we use

$$E(k, S, V_1 - \delta V_2) \geq (1 - \delta)E(k, S, V_1) + \delta E(k, S, V_1 - V_2). \tag{2.1}$$

We have $E(k, S, V_1 - V_2) \geq -Bk$ due to the stability of the potential, and

$$E(k, S, V_1) \geq \left\lfloor \frac{k}{2} \right\rfloor E(2, S, V_1) \geq \frac{k}{3} E(2, S, V_1) \tag{2.2}$$

from the superadditivity. These show that $E(k, S, V_1 - \delta V_2) \geq 0$ if we let δ small enough, which depends on ℓ.

Step 2: *Non-negativity of the ground state energy in any large boxes. (Theorem 3.1.)*

We divide the large box into small boxes. For a positive potential case, the theorem is trivial, since the energy decreases if we neglect interaction between particles in different boxes. In the negative potential case, however, this may increase energy.

To resolve the problem, we first assume that we have a torus and change the origin of division continuously, hence consider a set of divisions. In each choice of a division, we find the sum of energy in small boxes, and then calculate the average of this sum for the whole set of division. Then, if the size of the small box is much larger than the range of the potential, an interaction between two particles is neglected only in a very small portion of possible divisions. This means that this average can be a good estimate for the energy in the torus. By adjusting the negative potential a little bit, we can prove the non-negativity of the ground state energy in any large tori.

To prove the theorem for a box with Neumann boundary conditions, we notice that the only difference between a torus and a box comes from that, when we move the origin of the division, we have more smaller boxes, whose sizes vary, at the boundary of the large box for a box case. This does not cause any problem, however, because the ground state energy is non-negative whenever the box is small enough to be contained in the small box Λ_ℓ.

Step 3: *Lower bound for the ground state energy of two particles. (Lemma 4.2.)*
We first estimate the ground state energy of Neumann problem (Lemma 4.1)

$$\left(-\Delta + \frac{1}{2}V\right)\phi = E\phi \tag{2.3}$$

on a sphere of radius ℓ_0. Using the perturbation theory (Lemma 3.2), we find a lower bound for the ground state energy of two particles.

Step 4: *Dividing and subdividing $\mathbb{T}$ into small boxes. (Lemma 4.3.)*
We divide $\mathbb{T}$ into small boxes of side length ℓ_2 and find a lower bound for the ground state energy in the small box Λ_{ℓ_2}. To find the lower bound, we subdivide the small box into smaller boxes of side length $\ell_1 \gg \ell_0$, Λ_{ℓ_1}. Note that these division and subdivision include the technique of changing origin in Step 2.

We let ℓ_1 small so that average number of particles in Λ_{ℓ_1} is much less than one. Furthermore, we neglect the energy in Λ_{ℓ_1} whenever Λ_{ℓ_1} contains three or more particles; this does not increase the total energy in Λ_{ℓ_2} due to the non-negativity of the ground state energy (Theorem 3.1). To actually find the lower bound, we use the perturbation theory and estimate the number of cases in which exactly two particles are in the same Λ_{ℓ_1}, when using the constant function, which is the ground state of the unperturbed Hamiltonian.

Step 5: *Lower bound of the ground state energy in $\mathbb{T}$. (Theorem 1.3, Main Theorem.)*
Using superadditivity, we prove that we get the lower bound when the particles are evenly distributed among the small boxes, Λ_{ℓ_2}. The actual calculation gives the lower bound of the main theorem.

3 Nonnegativity of Ground State Energy

In this section, we prove the following theorem.

Theorem 3.1 *For any sufficiently large ℓ', there exists a positive constant c_1 such that the ground state energy of* (1.2), *where $V = V_1 - c_1 V_2$ satisfying the assumptions of Theorem* 1.3, *in $\mathbb{T}_{\ell'}$, a three dimensional torus of size ℓ', or in $\Lambda_{\ell'}$, a box of size ℓ', is non-negative for any n, i.e.*

$$E(n, \mathbb{T}_{\ell'}, V_1 - c_1 V_2) \geq 0, \tag{3.1}$$

and

$$E(n, \Lambda_{\ell'}, V_1 - c_1 V_2) \geq 0. \tag{3.2}$$

We first prove this for a fixed small cell Λ_ℓ of side length ℓ, which contains k particles. To begin with, we first show a lemma that we will use throughout this paper.

Lemma 3.2 *Let A be a non-negative Hermitian operator on $L^2(S)$, where S is a bounded set. Assume that the constant function ψ_0 is an eigenfunction of A with a simple eigenvalue* 0 *and* $\|\psi_0\|_{L^2(S)} = 1$. *Furthermore, assume that the second smallest eigenvalue, the gap, of A is γ. Suppose that X is a multiplication operator on $L^2(S)$ and let $X_\infty = \|X\|_{L^\infty(S)}$. Then, if $\gamma \geq 4X_\infty$,*

$$\inf \operatorname{spec}\,(A + X) \geq \langle \psi_0, X\psi_0 \rangle - \frac{2|X_\infty|^2}{\gamma}. \tag{3.3}$$

Proof of Lemma 3.2 Let $E_0 = \inf \operatorname{spec}\,(A + X)$. Then,

$$E_0 \leq \langle \psi_0, (A + X)\psi_0 \rangle = \langle \psi_0, X\psi_0 \rangle \leq X_\infty. \tag{3.4}$$

Let $\psi_0 + \psi'$ be the eigenfunction of $A + X$ with the smallest eigenvalue and $\langle \psi_0, \psi' \rangle = 0$. Then,

$$(A + X)(\psi_0 + \psi') = E_0(\psi_0 + \psi'). \tag{3.5}$$

Taking inner products of both sides of (3.5) with ψ_0 and ψ' gives

$$\langle \psi_0, X(\psi_0 + \psi') \rangle = E_0, \tag{3.6}$$

$$\langle \psi', X(\psi_0 + \psi') \rangle + \langle \psi', A\psi' \rangle = E_0 \langle \psi', \psi' \rangle, \tag{3.7}$$

respectively. By the assumption, $\langle \psi', A\psi' \rangle \geq \gamma \langle \psi', \psi' \rangle$, thus

$$E_0 \langle \psi', \psi' \rangle \geq \langle \psi', X(\psi_0 + \psi') \rangle + \gamma \langle \psi', \psi' \rangle. \tag{3.8}$$

From the Schwarz inequality, we have

$$\begin{aligned}\langle \psi', X(\psi_0 + \psi') \rangle &\geq -X_\infty \|\psi'\|_{L^2(S)} \|\psi_0 + \psi'\|_{L^2(S)} = -X_\infty \|\psi'\|_{L^2(S)} (1 + \|\psi'\|^2_{L^2(S)})^{\frac{1}{2}} \\ &\geq -X_\infty (\|\psi'\|_{L^2(S)} + \|\psi'\|^2_{L^2(S)}).\end{aligned} \tag{3.9}$$

Thus, from (3.8) and (3.9),

$$(E_0 - \gamma)\|\psi'\|^2_{L^2(S)} \geq -X_\infty (\|\psi'\|_{L^2(S)} + \|\psi'\|^2_{L^2(S)}), \tag{3.10}$$

and, since $E_0 \leq X_\infty \leq \gamma$,

$$\|\psi'\|_{L^2(S)} \leq \frac{X_\infty}{\gamma - E_0 - X_\infty}. \tag{3.11}$$

Therefore, from (3.6),

$$\begin{aligned} E_0 &= \langle\psi_0, X\psi_0\rangle + \langle\psi_0, X\psi'\rangle \\ &\geq \langle\psi_0, X\psi_0\rangle - X_\infty\|\psi'\|_{L^2(B)} \geq \langle\psi_0, X\psi_0\rangle - \frac{|X_\infty|^2}{\gamma - E_0 - X_\infty} \\ &\geq \langle\psi_0, X\psi_0\rangle - \frac{2|X_\infty|^2}{\gamma}, \end{aligned} \tag{3.12}$$

which was to be proved. □

Now we prove Theorem 3.1 for a small box Λ_ℓ of side length ℓ, which contains k particles.

Lemma 3.3 *Let Λ_ℓ be a box of side length ℓ. Then, there exists $\ell > R_1$ such that, there exists $\delta > 0$ such that for any rectangular box $S \subset \Lambda_\ell$ and for any $k \geq 2$, the ground state energy of*

$$H^{(k)} = -\sum_{j=1}^{k} \Delta_j + \sum_{i<j}^{k} (V_1 - \delta V_2)(x_i - x_j) \tag{3.13}$$

in S with Neumann boundary conditions is non-negative, i.e.

$$E(k, S, V_1 - \delta V_2) \geq 0. \tag{3.14}$$

Here, R_1 is the range of the potential $V_1 - \delta V_2$ as in Theorem 1.3.

Proof of Lemma 3.3 We first prove the lemma when $k = 2$. Since $V_1 - V_2$ is stable, $V_1(0) > 0$. Thus, we can find $R > 0$ such that $V_1(x) > \frac{V_1(0)}{2}$ if $|x| < R$. Let

$$V_1'(x) = \begin{cases} V_1(x) - \frac{V_1(0)}{2} & \text{if } |x| < R, \\ 0 & \text{otherwise} \end{cases} \tag{3.15}$$

and a' be the scattering length of $2V_1'$.

By change of variable $\xi = x_1 + x_2$, $\eta = x_1 - x_2$, and $S' = \{(\xi, \eta) | \frac{\xi+\eta}{2}, \frac{\xi-\eta}{2} \in \Lambda_\ell\}$, we get

$$\begin{aligned} &\int_{S^2} dx_1 dx_2 \big[|\nabla_1\psi|^2 + |\nabla_2\psi|^2 + V_1(x_1 - x_2)|\psi|^2\big] \\ &\quad = \frac{1}{8}\int_{S^2} d\xi d\eta \big[|\nabla_\xi\psi|^2 + |\nabla_\eta\psi|^2 + V_1(\eta)|\psi|^2\big] \\ &\quad \geq \frac{1}{8}\int_{S'} d\xi d\eta \bigg[|\nabla_\eta\psi|^2 + V_1'(\eta)|\psi|^2 + \frac{V_1(0)}{2} 1(|\eta| < R)|\psi|^2\bigg]. \end{aligned} \tag{3.16}$$

To find the lower bound, we use the following lemma, the generalization of a Lemma of Dyson [4].

Lemma 3.4 *Let $v(r) \geq 0$ with scattering length a and $v(r) = 0$ for $r > R_0$. Let $U(r) \geq 0$ be any function satisfying $\int U(r) r^2 dr \leq 1$ and $U(r) = 0$ for $r < R_0$. Let $D \subset \mathbb{R}^3$ be star-*

shaped (convex suffices) with respect to 0. *Then, for all differentiable functions* ϕ,

$$\int_D dx\Big[|\nabla\phi(x)|^2+\frac{1}{2}v(r)|\phi(x)|^2\Big]\geq a\int_D dxU(r)|\phi(x)|^2. \tag{3.17}$$

Proof See Lemma 1 in [4]. □

For any fixed ξ, let $S_\xi=\{\eta|\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\in\Lambda_\ell\}$. If we let

$$U(r)=\begin{cases}3[(2\ell)^3-R^3]^{-1} & \text{if } R<r<2\ell,\\ 0 & \text{otherwise,}\end{cases} \tag{3.18}$$

then $\int U(r)r^2dr=1$. By Lemma 3.4, since S_ξ is convex,

$$\int_{S_\xi} d\eta\big[|\nabla_\eta\psi|^2+V_1'(\eta)|\psi|^2\big]\geq a'\int_{S_\xi} d\eta U(\eta)|\psi|^2. \tag{3.19}$$

Thus, if we choose ℓ large so that

$$\frac{3a'}{(2\ell)^3-R^3}<\frac{V_1(0)}{2}, \tag{3.20}$$

then we get

$$\int_{S_\xi} d\eta\Big[|\nabla_\eta\psi|^2+V_1'(\eta)|\psi|^2+\frac{V_1(0)}{2}1(|\eta|<R)|\psi|^2\Big]\geq\frac{3a'}{(2\ell)^3-R^3}\int_{S_\xi} d\eta|\psi|^2. \tag{3.21}$$

Hence,

$$E(2,S,V_1)=\inf_{\|\psi\|_2=1}\int_{S^2}dx_1dx_2\Big[|\nabla_1\psi|^2+|\nabla_2\psi|^2+V_1(x_1-x_2)|\psi|^2\Big]\geq\frac{3a'}{(2\ell)^3-R^3}. \tag{3.22}$$

Now, from the superadditivity (1.5),

$$E(k,S,V_1)\geq\Big\lfloor\frac{k}{2}\Big\rfloor E(2,S,V_1)\geq\frac{k}{3}E(2,S,V_1)\geq\frac{a'k}{(2\ell)^3-R^3}, \tag{3.23}$$

where $\lfloor\frac{k}{2}\rfloor$ denotes the greatest integer that does not exceed $\frac{k}{2}$. Hence, if

$$\delta\leq\min\Big\{\frac{1}{2},\frac{a'}{2B[(2\ell)^3-R^3]}\Big\}, \tag{3.24}$$

then,

$$\begin{aligned}&\sum_{j=1}^{k}\int_{S^k}dX_k|\nabla_j\psi(X_k)|^2+\sum_{i<j}^{k}\int_{S^k}dX_k\big(V_1(x_i-x_j)-\delta V_2(x_i-x_j)\big)|\psi(X_k)|^2\\&\quad\geq(1-\delta)\Bigg[\sum_{j=1}^{k}\int_{S^k}dX_k|\nabla_j\psi(X_k)|^2+\sum_{i<j}^{k}\int_{S^k}dX_kV_1(x_i-x_j)|\psi(X_k)|^2\Bigg]\end{aligned}$$

$$+\delta\sum_{i<j}^{k}\int_{S^k} dX_k\big(V_1(x_i-x_j)-V_2(x_i-x_j)\big)|\psi(X_k)|^2$$

$$\geq \frac{1}{2}\cdot\frac{a'k}{(2\ell)^3-R^3}-\delta Bk\geq 0. \tag{3.25}$$

This proves the lemma. □

Lemma 3.3 proves Theorem 3.1 when ℓ' is not too big. Since δ depends on ℓ, however, the argument we used in the proof of Lemma 3.3 is no longer valid for a fixed δ when ℓ' is large.

In the case when ℓ' is large, we divide the box of size ℓ' into smaller boxes. However, the potential here contains small negative parts in it, so that dividing the box does not guarantee the ground state energy increasing. Thus, using the argument similar to [1], we change the grid for division continuously and take an average of the ground state energy for such divisions.

Proof of Theorem 3.1 We first prove the torus case. Let χ be a characteristic function of the unit cube,

$$\chi(x)=\begin{cases}1 & x\in[0,1]^3,\\ 0 & \text{otherwise}\end{cases} \tag{3.26}$$

defined on $\mathbb{T}_{\ell'}=\mathbb{R}^3/(\ell'\mathbb{Z})^3$. Here, ℓ satisfies Lemma 3.3 and $\ell'\gg\ell$.

Let $\chi_{u\lambda}(x)=\chi(x+u+\lambda)$. Here, $\lambda\in G$ where $G=\{p\in\mathbb{Z}^3|\ell p\in\Lambda_{\ell'}\}$, and $u\in[0,1]^3\equiv\Gamma$. Then, $\sum_{\lambda\in G}\chi_{u\lambda}(x)=1$ for all $x\in\mathbb{T}_{\ell'}$, $u\in\Gamma$.

A function h is defined by

$$h(x,y)=\int_\Gamma du\sum_{\lambda\in G}\chi_{u\lambda}(x)\chi_{u\lambda}(y). \tag{3.27}$$

Then, h depends only on the difference $z=x-y$ and we can let

$$h(z)=h(x,y)=\int_\Gamma du\sum_{\lambda\in G}\chi(x+u+\lambda)\chi(y+u+\lambda)$$

$$=\int_{\mathbb{T}_{\ell'}} du\,\chi(x+u)\chi(y+u)=(\chi*\chi)(z). \tag{3.28}$$

Let $h_\ell(x)=h(x/\ell)$. We can define localized kinetic and potential energies. Let $\alpha=(u,\alpha_1,\dots,\alpha_n)\in\Gamma\times G^n$ be a multi-index and $\int d\alpha=\int_\Gamma du\sum_{\alpha_1\in G}\cdots\sum_{\alpha_n\in G}$. Let

$$\psi_\alpha^\ell(x_1,\dots,x_n)=\prod_{k=1}^n\chi_{u\alpha_k}\left(\frac{x_k}{\ell}\right)\psi(x_1,\dots,x_n), \tag{3.29}$$

$$V_\alpha=\sum_{i<j}^n\delta_{\alpha_i\alpha_j}V, \tag{3.30}$$

where $V = V_1 - \delta V_2$. Then from the definition of χ and h, we have

$$\begin{aligned}\int d\alpha V_\alpha |\psi_\alpha^\ell|^2 &= \sum_{i<j}^n \int_{\mathbb{T}_{\ell'}^n} dX_n \int d\alpha \prod_{k=1}^n \chi_{u\alpha_k}\left(\frac{x_k}{\ell}\right) V(x_i - x_j)\delta_{\alpha_i\alpha_j}|\psi(X_n)|^2 \\ &= \sum_{i<j}^n \int_{\mathbb{T}_{\ell'}^n} dX_n V(x_i - x_j) h_\ell(x_i - x_j)|\psi(X_n)|^2. \end{aligned} \tag{3.31}$$

We also know that

$$\begin{aligned} &\sum_{j=1}^n \int_{\mathbb{T}_{\ell'}^n} dX_n |\nabla_j \psi(X_n)|^2 \\ &= \int_\Gamma du \sum_{\sum n_\sigma = n} \int_{\mathbb{T}_{\ell'}^n} dX_n \sum_{j=1}^n |\nabla_j \psi(X_n)|^2 \prod_\sigma 1(N_{u\sigma}(X_n) = n_\sigma) \\ &= \sum_{\sum N_\sigma = n} \int d\alpha \int_{\mathbb{T}_{\ell'}^n} dX_n \prod_{k=1}^n \chi_{u\alpha_k}\left(\frac{x_k}{\ell}\right) \sum_{j=1}^n |\nabla_j \psi(X_n)|^2 \prod_\sigma 1(N_{u\sigma}(X_n) = n_\sigma) \end{aligned} \tag{3.32}$$

where $N_{u\sigma}(X_n) = \sum_{j=1}^n \chi_{u\sigma}(x_j)$.

Thus, from the equations above, we get the following lemma.

Lemma 3.5 *For any n,*

$$E(n, \mathbb{T}_{\ell'}, V h_\ell) \geq \inf_{\sum n_\sigma = n} \sum_{\sigma \in G} E(n_\sigma, \Lambda_\ell, V). \tag{3.33}$$

Proof of Lemma 3.5

$$\begin{aligned} &\sum_{j=1}^n \int_{\mathbb{T}_{\ell'}^n} dX_n |\nabla_j \psi(X_n)|^2 + \sum_{i<j}^n \int_{\mathbb{T}_{\ell'}^n} dX_n\, V(x_i - x_j) h_\ell(x_i - x_j)|\psi(X_n)|^2 \\ &\geq \left(\int d\alpha \|\psi_\alpha^\ell\|_2^2\right) \inf_{\sum n_\sigma = n} \sum_{\sigma \in G} E(n_\sigma, \Lambda_\ell, V) = \inf_{\sum n_\sigma = n} \sum_{\sigma\in G} E(n_\sigma, \Lambda_\ell, V). \end{aligned} \tag{3.34}$$

□

Choose R_1 such that $V_2(x_i - x_j) = 0$ when $|x_i - x_j| \geq R_1$. Since $R_1 = O(1)$ and $h(z) = (\chi * \chi)(z) = g(z^1)g(z^2)g(z^3)$ where $z = (z^1, z^2, z^3)$ and

$$g(t) = \begin{cases} 1 - |t| & |t| \leq 1, \\ 0 & \text{otherwise} \end{cases} \tag{3.35}$$

we can let $\ell \geq R_1$ so that $|x_i - x_j| < R_1$ implies $h_\ell(x_i - x_j) \geq 1 - \frac{\sqrt{3}R_1}{\ell}$. Now, if we let

$$c_1 = \left(1 - \frac{\sqrt{3}R_1}{\ell}\right)\delta, \tag{3.36}$$

with δ satisfying Lemma 3.3, then $V_1 \geq h_\ell V_1$ and $c_1 V_2 \leq \delta V_2 h_\ell$, hence

$$\begin{aligned}\sum_{j=1}^{n} \int_{\mathbb{T}_{\ell'}} dX_n |\nabla_j \psi(X_n)|^2 + \sum_{i<j}^{n} \int_{\mathbb{T}_{\ell'}} dX_n\, (V_1 - c_1 V_2)(x_i - x_j)|\psi(X_n)|^2 \\ \geq \sum_{j=1}^{n} \int_{\mathbb{T}_{\ell'}} dX_n |\nabla_j \psi(X_n)|^2 \\ + \sum_{i<j}^{n} \int_{\mathbb{T}_{\ell'}} dX_n\, (V_1 - \delta V_2)(x_i - x_j) h_\ell (x_i - x_j)|\psi(X_n)|^2. \end{aligned} \tag{3.37}$$

Hence, if $E(n_\sigma, \Lambda_\ell, V) \geq 0$ for any n_σ, the ground state energy is non-negative, and it is already proved in Lemma 3.3. This proves the first part of the theorem.

The box case can be proved in a similar way. When we have a box of size ℓ', $\Lambda_{\ell'}$, the only difference between $\Lambda_{\ell'}$ and $\mathbb{T}_{\ell'}$ is that $\Lambda_{\ell'}$ has walls at the boundary, in the sense that the particles do not interact across these walls. Thus, if we use the same argument as in the case of periodic boundary conditions, we can get the Lemma 3.5 with some small cells at the boundary of $\Lambda_{\ell'}$ having walls in them. If one of these small cells, B, is divided into $B_1, B_2, \dots, B_m$ by the walls, then, in this case, $E(k, B, V)$ actually denotes $\inf_{\sum k_m = k} \sum_{j=1}^{m} E(k_j, B_j, V)$.

Each $E(k_j, B_j, V)$ is the ground state energy in B_j, which is smaller than B, and Lemma 3.3 holds for smaller cells. Thus, $E(k_j, B_j, V) \geq 0$ and $\inf_{\sum k_m = k} \sum_{j=1}^{m} E(k_j, B_j, V) \geq 0$. Hence, we get the desired theorem. □

4 Lower Bound of Ground State Energy

With the aid of the Theorem 3.1, we will prove our main result. Assuming conditions in Theorem 1.3, we start with a lemma which is similar to Lemma A.1 in [3], with the interaction potential

$$V = V_1 - \frac{1}{2} c_1 V_2 = V_1 - \lambda V_2, \tag{4.1}$$

where c_1 satisfies Theorem 3.1 and is defined by (3.24) and (3.36).

Lemma 4.1 *Let ϕ be a solution of Neumann Problem*

$$\left(-\Delta + \frac{1}{2} V\right)\phi = E\phi \tag{4.2}$$

on the sphere of radius ℓ_0, where E is the ground state energy, with the boundary condition

$$\phi(\ell_0) = 1, \qquad \frac{\partial \phi}{\partial r}(\ell_0) = 0. \tag{4.3}$$

Let a be the scattering length of V. Then if $\ell_0 \gg R_1$, we have

$$E \geq \frac{3a}{\ell_0^3}\left(1 + O\left(\frac{1}{\ell_0}\right)\right). \tag{4.4}$$

Proof of Lemma 4.1 Let w be the solution of zero-energy scattering equation,

$$\left(-\Delta+\frac{1}{2}V\right)w=0, \qquad \lim_{r\to\infty} w(r)=1. \tag{4.5}$$

Then, by definition, $f(r):=rw(r)=r-a$ for $r>R_1$ where $r=|x|$. Let

$$\psi(r)=\frac{\sin hf(r)}{r} \tag{4.6}$$

if $r<\ell_0$ and $\psi(r)=\frac{\sin hf(\ell_0)}{\ell_0}$ otherwise, where h is the smallest positive number satisfying $\psi'(\ell_0)=0$. It gives $h\ell_0=\tan hf(\ell_0)$, or

$$h^2=\frac{3a}{\ell_0^3}+O\left(\frac{1}{\ell_0^4}\right). \tag{4.7}$$

Let $\phi=\psi g$. Then

$$\int_{r<\ell_0}\overline{\phi}\left(-\Delta+\frac{1}{2}V\right)\phi=\int_{r<\ell_0}|g|^2\psi\left(-\Delta+\frac{1}{2}V\right)\psi+\int_{r<\ell_0}|\nabla g|^2\psi^2. \tag{4.8}$$

A calculation shows that

$$\begin{aligned}
&\psi\left(-\Delta+\frac{1}{2}V\right)\psi\\
&=\frac{1}{2}V\psi^2+\psi\left[\frac{h^2\{f'(r)\}^2\sin hf(r)}{r}-\frac{hf''(r)\cos hf(r)}{r}\right]\\
&=h^2\psi^2+h^2\psi^2(\{f'(r)\}^2-1)\\
&\quad-\frac{1}{r^2}\left[hf''(r)\cos hf(r)\sin hf(r)+\frac{1}{2}V\sin^2 hf(r)\right].
\end{aligned} \tag{4.9}$$

The last two terms vanish where $r>R_1$. Furthermore, in the Taylor expansion with respect to h, $O(h^2)$ terms cancel in the square bracket, since $-f''(r)+\frac{1}{2}V(r)f(r)=0$. Thus, we have

$$\begin{aligned}
&\int_{r<\ell_0}\overline{\phi}\left(-\Delta+\frac{1}{2}V\right)\phi\\
&\quad\ge h^2\int_{r<\ell_0}|\phi|^2+h^2\int_{r<\ell_0}(\{f'(r)\}^2-1)|\phi|^2\\
&\quad\quad-Ch^3\int_{r<R_1}\frac{|g|^2}{r^2}+\int_{r<\ell_0}|\nabla g|^2\psi^2.
\end{aligned} \tag{4.10}$$

Since $f'(r)=1$ for $r>R_1$ and f' is bounded, we can see that $\int(f'(r)^2-1)|\phi|^2/\int|\phi|^2=O(\ell_0^{-1})$ as $\ell_0\to\infty$. We also know that $\sin hf(r)\ge Chr$ and f does not vanish. Hence, $\psi\ge Ch$ and

$$\int_{r<\ell_0}|\nabla g|^2\psi^2\ge Ch^2\int_{r<\ell_0}|\nabla g|^2. \tag{4.11}$$

Now, from the Hardy type inequality (Lemma 5.1 in [3]),

$$\int_{r<\ell_0} \frac{1(|x|<R_1)}{r^2}|g|^2 \le C\int_{r<\ell_0}|\nabla g|^2 + C\frac{R_1^3}{\ell_0^3}\int_{r<\ell_0}|g|^2, \tag{4.12}$$

where we used

$$|\{x\in\mathbb{R}^3 : |x|<\ell_0\}|^{-1}\int_{r<\ell_0} 1(|x|<R_1) = \frac{R_1^3}{\ell_0^3}. \tag{4.13}$$

Thus,

$$\begin{aligned}\int_{r<\ell_0}|\nabla g|^2\psi^2 - Ch^3\int_{r<R_1}\frac{|g|^2}{r^2} &\ge C\left(h^3\int_{r<\ell_0}|\nabla g|^2 - h^3\int_{r<R_1}\frac{|g|^2}{r^2}\right)\\ &\ge -Ch^3\frac{R_1^3}{\ell_0^3}\int_{r<\ell_0}|g|^2.\end{aligned} \tag{4.14}$$

Since $h^2 = O(\ell_0^{-3})$ and $\psi^2 = O(\ell_0^{-3})$, the last term is bounded by $Ch^3\int|\phi|^2$, and we get the lemma. $\square$

To prove the two-particle case, let $\ell_1^3 = \rho^{-1+\epsilon}$, $\ell_0^3 = \rho^{-1+5\epsilon}$, $\epsilon < \frac{1}{31}$. This choice of ℓ_1 guarantees Λ_{ℓ_1}, a box of side length $\ell_1 \gg \ell_0$, contains almost no particles in average.

Lemma 4.2 *Let the two-particle Hamiltonian*

$$H^{(2)} = -\Delta_1 - \Delta_2 + V(x_1 - x_2). \tag{4.15}$$

Let the density of particles in the thermodynamic limit $\rho = N/L^3$, and $\ell_1^3 = \rho^{-1+\epsilon}$. Then, there exists a positive constant ϵ such that its ground state energy with Neumann boundary conditions satisfies

$$E(2,\Lambda_{\ell_1},V) \ge (1-\rho^\epsilon)\frac{8\pi a}{\ell_1^3}. \tag{4.16}$$

Proof of Lemma 4.2 For two-particle bosonic function $\phi(x_1,x_2)$ with Neumann boundary condition, we have

$$\begin{aligned}&\int_{\Lambda_{\ell_1}^2} dx_1dx_2\overline{\phi}H^{(2)}\phi\\ &\quad= \frac{\rho^\epsilon}{2}\int_{\Lambda_{\ell_1}^2} dx_1dx_2\overline{\phi}(-\Delta_1-\Delta_2)\phi + (1-\rho^\epsilon)\\ &\qquad\times\int_{\Lambda_{\ell_1}^2} dx_1dx_2\overline{\phi}\left(-\Delta_1 + \frac{1}{2}V(x_1-x_2)\right)\phi\\ &\qquad+(1-\rho^\epsilon)\int_{\Lambda_{\ell_1}^2} dx_1dx_2\overline{\phi}\left(-\Delta_2 + \frac{1}{2}V(x_1-x_2)\right)\phi\\ &\qquad+\frac{\rho^\epsilon}{2}\int_{\Lambda_{\ell_1}^2} dx_1dx_2\overline{\phi}\big((-\Delta_1-\Delta_2)+2V(x_1-x_2)\big)\phi.\end{aligned} \tag{4.17}$$

For $j=1,2$, from Lemma 4.1,

$$\int_{\Lambda_{\ell_1}^2} dx_1 dx_2 \overline{\phi}\Big(-\Delta_j + \frac{1}{2}V(x_1-x_2)\Big)\phi \\ \geq \frac{3a}{\ell_0^3}\left(1+O\left(\frac{1}{\ell_0}\right)\right)\int_{\Lambda_{\ell_1}^2} dx_1 dx_2 1(|x_1-x_2|<\ell_0)|\phi|^2, \tag{4.18}$$

where $\ell_0^3=\rho^{-1+5\epsilon}$. From Lemma 3.3, the last term in (4.17) is non-negative, since

$$\int_{\Lambda_{\ell_1}^2} dx_1 dx_2 \overline{\phi}\big((-\Delta_1-\Delta_2)+2V(x_1-x_2)\big)\phi \\ \geq \int_{\Lambda_{\ell_1}^2} dx_1 dx_2 \overline{\phi}\big((-\Delta_1-\Delta_2)+(V_1-c_1V_2)(x_1-x_2)\big)\phi \geq 0. \tag{4.19}$$

This proves

$$H^{(2)} \geq H_0 + H_1 \tag{4.20}$$

where

$$H_0 = \frac{\rho^\epsilon}{2}(-\Delta_1-\Delta_2) \tag{4.21}$$

and

$$H_1 = \frac{6a}{\ell_0^3}\left(1+O\left(\frac{1}{\ell_0}\right)\right)(1-\rho^\epsilon)1(|x_1-x_2|\leq \ell_0). \tag{4.22}$$

We let H_0 be the unperturbed part and H_1 the perturbation in H_0+H_1. Lemma 3.2 tells that the ground state energy of $H^{(2)}$ satisfies

$$E(2,\Lambda_{\ell_1},V) \geq \int_{\Lambda_{\ell_1}^2} dx_1 dx_2 \overline{\psi_0} H_1 \psi_0 - \frac{2\|H_1\|_\infty^2}{\gamma} \tag{4.23}$$

where ψ_0 is the ground state of H_0, and $\gamma = C\rho^\epsilon \ell_1^{-2}$ is the spectral gap of H_0. We need the condition $\gamma \geq \|H_1\|_\infty$ and it indeed is true, since

$$\gamma = C\rho^\epsilon \ell_1^{-2} \gg \frac{6a}{\ell_0^3} \tag{4.24}$$

if $\epsilon < \frac{1}{16}$.

Letting $\psi_0 = \ell_1^{-3}$,

$$E(2,\Lambda_{\ell_1},V) \geq \frac{6a}{\ell_0^3}(1-\rho^\epsilon)\int_{\Lambda_{\ell_1}^2} dx_1 dx_2 \ell_1^{-6} 1(|x_1-x_2|\leq\ell_0) - \frac{2\|H_1\|_\infty^2}{\gamma} \\ \geq \frac{6a}{\ell_0^3}(1-\rho^\epsilon)\frac{4\pi\ell_0^3}{3}\frac{1}{\ell_1^3} = (1-\rho^\epsilon)\frac{8\pi a}{\ell_1^3} \tag{4.25}$$

up to higher order terms of ρ. Note that

$$\frac{\|H_1\|_\infty^2}{\gamma} \ll \frac{\rho^\epsilon}{\ell_1^3} \tag{4.26}$$

if $\epsilon < \frac{1}{31}$. This proves Lemma 4.2. □

To prove Theorem 1.3, we divide $\mathbb{T}$ into small boxes of side length ℓ_2, each of which is subdivided into smaller boxes of side length ℓ_1. Let $\kappa = \rho^{-\epsilon}$, $\ell_2 = \ell_1\kappa = \rho^{-\frac{1}{3}-\frac{2}{3}\epsilon}$. This means $\kappa^2 = N(\ell_2^3/L^3)$, i.e., the typical number of particles in Λ_{ℓ_2}, a box of side length ℓ_2, is κ^2. When the number of particles in a box is not too large, we can get the following lower bound.

Lemma 4.3 *Let $V = V_1 - \lambda V_2$ and a_1 be the scattering length of V_1. Suppose that $k \le M = \frac{a}{a_1}\cdot 8\kappa^2 + 1$. Given the Hamiltonian of k-particle system in a box of side length ℓ_2*

$$H^{(k)} = -\sum_{j=1}^{k} \Delta_j + \sum_{i<j}^{k} V(x_i - x_j), \tag{4.27}$$

there exist positive constants C' and ϵ such that, the ground state energy of (4.27) with Neumann boundary conditions,

$$E(k, \Lambda_{\ell_2}, V) \ge \frac{4\pi a}{\ell_2^3} k(k-1)(1 - C'\rho^{\epsilon}). \tag{4.28}$$

Proof of Lemma 4.3 We have

$$\begin{aligned}
&(1+5\rho^{\epsilon})\left[(1-\rho^{\epsilon})\sum_{j=1}^{k}\int_{\Lambda_{\ell_2}^k} dX_k |\nabla_j \psi(X_k)|^2 + \sum_{i<j}^{k}\int_{\Lambda_{\ell_2}^k} dX_k V(x_i - x_j)|\psi(X_k)|^2\right] \\
&\ge \left[\sum_{j=1}^{k}\int_{\Lambda_{\ell_2}^k} dX_k |\nabla_j \psi(X_k)|^2 + \sum_{i<j}^{k}\int_{\Lambda_{\ell_2}^k} dX_k (V_1 - (1-\rho^{\epsilon})\lambda V_2)(x_i - x_j)|\psi(X_k)|^2\right] \\
&\quad + 3\rho^{\epsilon}\left[\sum_{j=1}^{k}\int_{\Lambda_{\ell_2}^k} dX_k |\nabla_j \psi(X_k)|^2 + \sum_{i<j}^{k}\int_{\Lambda_{\ell_2}^k} dX_k (V_1 - 2\lambda V_2)(x_i - x_j)|\psi(X_k)|^2\right].
\end{aligned} \tag{4.29}$$

From Theorem 3.1, we know that the second term in the right hand side of (4.29) is non-negative. Subdivide Λ_{ℓ_2} into smaller boxes of side length ℓ_1 as in the proof of Theorem 3.1. Since $1 - \rho^{\epsilon} \le 1 - \frac{\sqrt{3}R_1}{\ell_1} \le h_{\ell_1}$, from Lemma 3.5,

$$\begin{aligned}
&\sum_{j=1}^{k}\int_{\Lambda_{\ell_2}^k} dX_k |\nabla_j \psi(X_k)|^2 + \sum_{i<j}^{k}\int_{\Lambda_{\ell_2}^k} dX_k (V_1 - (1-\rho^{\epsilon})\lambda V_2)(x_i - x_j)|\psi(X_k)|^2 \\
&\quad \ge E(k, \Lambda_{\ell_2}, V h_{\ell_1}) \ge \inf_{u\in[0,1]^3} \inf_{\sum k_\sigma = k} \sum_{\sigma \in G'(u)} E(k_\sigma, \Lambda_{\ell_1}, V).
\end{aligned} \tag{4.30}$$

Here, u is the origin of a subdivision and $G'(u)$ is a set of small boxes in the subdivision that do not have walls in them as we already saw in the proof of Theorem 3.1. Note that Lemma 3.3 allows us to ignore small boxes that have walls.

Now, fix $u \in [0,1]^3$ and let β be an index for the small boxes in the subdivision that do not have walls in them. Let

$$q(n) = \begin{cases} \dfrac{1-\rho^\epsilon}{1+5\rho^\epsilon} \cdot \dfrac{8\pi a}{\ell_1^3} & \text{if } n = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.31}$$

If we let $N_\beta(X_k) = \sum_{j=1}^k 1(x_j \in \beta)$, then, when $N_\beta(X_k) \neq 2$, Theorem 3.1 shows that $E(N_\beta(X_k), \Lambda_{\ell_1}, V) \geq 0$, and when $N_\beta(X_k) = 2$, Lemma 4.2 shows that $E(N_\beta(X_k), \Lambda_{\ell_1}, V) \geq (1-\rho^\epsilon)\frac{8\pi a}{\ell_1^3}$. Thus, in any cases,

$$\begin{aligned} &\sum_{j=1}^k \int_{\Lambda_{\ell_2}^k} dX_k |\nabla_j \psi(X_k)|^2 + \sum_{i<j} \int_{\Lambda_{\ell_2}^k} dX_k (V_1 - (1-\rho^\epsilon)\lambda V_2)(x_i - x_j) |\psi(X_k)|^2 \\ &\quad \geq \sum_\beta q(N_\beta(X_k)). \end{aligned} \tag{4.32}$$

This shows that

$$H^{(k)} \geq H_1 = \rho^\epsilon \sum_{j=1}^k (-\Delta_j) + \sum_\beta q(N_\beta(X_k)) \tag{4.33}$$

with Neumann boundary conditions.

To estimate H_1, we let $W(X_k) = \sum_\beta q(N_\beta(X_k))$ be a perturbation. Since $W \leq \frac{k}{2} q(2)$, we have that γ, the spectral gap of the unperturbed Hamiltonian, $\rho^\epsilon \sum_{j=1}^k (-\Delta_j)$, satisfies

$$\gamma = C\rho^\epsilon \ell_2^{-2} \gg C\kappa^2 \ell_1^{-3} \geq \|W\|_\infty \tag{4.34}$$

if $\epsilon < \frac{1}{16}$. From Lemma 3.2, if we use the constant function $\ell_2^{-\frac{3}{2}k}$, the ground state of the unperturbed Hamiltonian, then

$$E(k, \Lambda_{\ell_2}, V) \geq \ell_2^{-3k} \int_{\Lambda_{\ell_2}^k} dX_k W(X_k) - C \frac{\|W\|_\infty^2}{\rho^\epsilon \ell_2^{-2}}. \tag{4.35}$$

To calculate the first term, for given β, we need to count the cases when exactly two particles are in β. We use the inclusion-exclusion principle to get

$$\begin{aligned} &\ell_2^{-3k} \int_{\Lambda_{\ell_2}^k} dX_k W(X_k) \\ &\quad \geq \ell_2^{-3k} \int_{\Lambda_{\ell_2}^k} dX_k \sum_\beta \sum_{i_1<i_2}^k 1(x_{i_1} \in \beta) 1(x_{i_2} \in \beta) \frac{1-\rho^\epsilon}{1+5\rho^\epsilon} \frac{8\pi a}{\ell_1^3} \\ &\qquad - 3\, \ell_2^{-3k} \int_{\Lambda_{\ell_2}^k} dX_k \sum_\beta \sum_{i_1<i_2<i_3}^k 1(x_{i_1} \in \beta) 1(x_{i_2} \in \beta) 1(x_{i_3} \in \beta) \frac{1-\rho^\epsilon}{1+5\rho^\epsilon} \frac{8\pi a}{\ell_1^3}. \end{aligned} \tag{4.36}$$

An explicit calculation shows

$$\ell_2^{-3k}\int_{\Lambda_{\ell_2}^k} dX_k W(X_k) \geq \frac{1-\rho^\epsilon}{1+5\rho^\epsilon}\frac{8\pi a}{\ell_1^3}\frac{k(k-1)}{2}\frac{\ell_1^3}{(\ell_2-\ell_1)^3} - 3\frac{1-\rho^\epsilon}{1+5\rho^\epsilon}\frac{8\pi a}{\ell_1^3}\frac{k(k-1)(k-2)}{6} \times \frac{\ell_1^3}{(\ell_2-\ell_1)^3}\frac{\ell_1^3}{\ell_2^3} \geq \frac{4\pi a}{\ell_2^3}k(k-1)(1-C\kappa^{-1}). \tag{4.37}$$

We want the last term in (4.35) is smaller than the error term in (4.28), $C'\frac{4\pi a}{\ell_2^3}k^2\rho^\epsilon$, and this holds if $\epsilon < \frac{1}{22}$, since

$$C\frac{\|W\|_\infty^2}{\rho^\epsilon \ell_2^{-2}} \leq C\frac{k^2\ell_2^2}{\rho^\epsilon \ell_1^6} \ll \frac{k^2\kappa^{-1}}{\ell_2^3} \tag{4.38}$$

provided $\epsilon < \frac{1}{22}$. Note that a does not depend on ρ hence could be absorbed into the constant C. This proves the lemma. $\square$

When $k > M$, from Theorem 3.1, we have, since $V = V_1 - \frac{1}{2}c_1 V_2 = V_1 - \lambda V_2$,

$$\begin{aligned}
&\sum_{j=1}^k \int_{\Lambda_{\ell_2}^k} dX_k |\nabla_j \psi(X_k)|^2 + \sum_{i<j}^k \int_{\Lambda_{\ell_2}^k} dX_k\, V(x_i-x_j)|\psi(X_k)|^2 \\
&= \frac{1}{2}\left[\sum_{j=1}^k \int_{\Lambda_{\ell_2}^k} dX_k |\nabla_j \psi(X_k)|^2 + \sum_{i<j}^k \int_{\Lambda_{\ell_2}^k} dX_k\, V_1(x_i-x_j)|\psi(X_k)|^2\right] \\
&\quad + \frac{1}{2}\left[\sum_{j=1}^k \int_{\Lambda_{\ell_2}^k} dX_k |\nabla_j \psi(X_k)|^2 + \sum_{i<j}^k \int_{\Lambda_{\ell_2}^k} dX_k \big((V_1 - c_1V_2)(x_i-x_j)\big)|\psi(X_k)|^2\right] \\
&\geq \frac{1}{2}\left[\sum_{j=1}^k \int_{\Lambda_{\ell_2}^k} dX_k |\nabla_j \psi(X_k)|^2 + \sum_{i<j}^k \int_{\Lambda_{\ell_2}^k} dX_k\, V_1(x_i-x_j)|\psi(X_k)|^2\right] \\
&\geq \frac{1}{2}E(k, \Lambda_{\ell_2}, V_1).
\end{aligned} \tag{4.39}$$

Now we prove the main theorem.

Proof of Theorem 1.3 Define λ by (3.24), (3.36), and (4.1). We have

$$\begin{aligned}
&\left(1+\frac{\sqrt{3}R_1}{\ell_2}\right)E(N, \mathbb{T}, V) \\
&\quad = \left(1+\frac{\sqrt{3}R_1}{\ell_2}\right)\inf_{\|\psi\|_2=1}\left[\sum_{j=1}^N \int_{\mathbb{T}^N} dX_N |\nabla_j \psi(X_N)|^2\right. \\
&\quad\quad \left. + \sum_{i<j}^N \int_{\mathbb{T}^N} dX_N V(x_i-x_j)|\psi(X_N)|^2\right]
\end{aligned}$$

$$
\begin{aligned}
&= \inf_{\|\psi\|_2=1}\Bigg[\sum_{j=1}^{N}\int_{\mathbb{T}^N} dX_N|\nabla_j\psi(X_N)|^2 \\
&\quad+\sum_{i<j}^{N}\int_{\mathbb{T}^N} dX_N\bigg(V_1-\bigg(1-\frac{\sqrt{3}R_1}{\ell_2}\bigg)\lambda V_2\bigg)(x_i-x_j)\big)|\psi(X_N)|^2 \\
&\quad+\frac{\sqrt{3}R_1}{\ell_2}\bigg(\sum_{j=1}^{N}\int_{\mathbb{T}^N} dX_N|\nabla_j\psi(X_N)|^2 \\
&\quad+\sum_{i<j}^{N}\int_{\mathbb{T}^N} dX_N(V_1-2\lambda V_2)|\psi(X_N)|^2\bigg)\Bigg].
\end{aligned} \tag{4.40}
$$

Here, R_1 denotes the range of V. Theorem 3.1 shows the last term in this equation satisfies

$$
\sum_{j=1}^{N}\int_{\mathbb{T}^N} dX_N|\nabla_j\psi(X_N)|^2+\sum_{i<j}^{N}\int_{\mathbb{T}^N} dX_N\big(V_1-2\lambda V_2\big)|\psi(X_N)|^2\geq 0. \tag{4.41}
$$

Hence,

$$
\bigg(1+\frac{\sqrt{3}R_1}{\ell_2}\bigg)E(N,\mathbb{T},V)\geq E\bigg(N,\mathbb{T},V_1-\bigg(1-\frac{\sqrt{3}R_1}{\ell_2}\bigg)\lambda V_2\bigg)\geq E\Big(N,\mathbb{T},\big(V_1-\lambda V_2\big)h_{\ell_2}\Big), \tag{4.42}
$$

where h_{ℓ_2} is defined by (3.27) using ℓ_2 instead of ℓ. Here, the second inequality follows from $1\geq h_{\ell_2}\geq (1-\frac{\sqrt{3}R_1}{\ell_2})$. Subdivide $\mathbb{T}$ into smaller boxes of side length ℓ_2 as in the proof of Theorem 3.1 and index them by β. From Lemma 3.5,

$$
E(N,\mathbb{T},Vh_{\ell_2})\geq \inf_{\sum k_\beta=N}\sum_{\beta}E(k_\beta,\Lambda_{\ell_2},V). \tag{4.43}
$$

Here, k_β denotes the number of particles in β.

From Lemma 4.3, if $k\leq M$, we have

$$
E(k,\Lambda_{\ell_2},V)\geq \frac{4\pi a}{\ell_2^3}k(k-1)(1-C'\rho^\epsilon). \tag{4.44}
$$

From the superadditivity (1.5), we get, if $k_\beta>M$,

$$
\begin{aligned}
E(k_\beta,\Lambda_{\ell_2},V_1)&\geq \left\lfloor\frac{k_\beta}{M}\right\rfloor E(M,\Lambda_{\ell_2},V_1)\geq \frac{k_\beta}{2M}E(M,\Lambda_{\ell_2},V_1)\\
&\geq \frac{4\pi a_1}{\ell_2^3}(1-C'\rho^\epsilon)\frac{k_\beta}{2M}M(M-1)=\frac{4\pi a}{\ell_2^3}(1-C'\rho^\epsilon)k_\beta\cdot 4\kappa^2,
\end{aligned} \tag{4.45}
$$

where $\lfloor\frac{k_\beta}{M}\rfloor$ denotes the greatest integer that does not exceed $\frac{k_\beta}{M}$.

Now we can actually calculate the lower bound using the argument similar to [4]. From (4.39), (4.44), and (4.45), we get

$$\inf_{\sum k_\beta = N} \sum_\beta E(k_\beta, \Lambda_{\ell_2}, V) \geq \frac{4\pi a}{\ell_2^3}(1 - C'\rho^\epsilon) \inf_{\sum k_\beta = N} \left[\sum_{\beta: k_\beta \leq M} k_\beta(k_\beta - 1) + \sum_{\beta: k_\beta > M} \frac{1}{2} k_\beta \cdot 4\kappa^2 \right]. \tag{4.46}$$

Let $t = \sum_{k_\beta \leq M} k_\beta$. Then,

$$\sum_{\beta: k_\beta \leq M} k_\beta^2 \geq \Bigg(\sum_{\beta: k_\beta \leq M} k_\beta \Bigg)^2 \Bigg/ \Bigg(\sum_{\beta: k_\beta \leq M} 1 \Bigg) \geq t^2 \Bigg/ \left(\frac{L^3}{\ell_2^3} \right), \tag{4.47}$$

hence $\sum k_\beta = N$ implies

$$\sum_{\beta: k_\beta \leq M} k_\beta(k_\beta - 1) + \sum_{\beta: k_\beta > M} \frac{1}{2} k_\beta \cdot 4\kappa^2 \geq \frac{\ell_2^3}{L^3} t^2 - t + 2\kappa^2(N - t) = \frac{\ell_2^3}{L^3}(t - N)^2 + N\kappa^2 - t. \tag{4.48}$$

Since $t \leq N$, the minimum of the right-hand side of (4.48) is attained when $t = N$. Therefore, from (4.42), (4.43), (4.46), and (4.48), there exists C_0 such that

$$E(N, \mathbb{T}, V) \geq \left(1 + \frac{\sqrt{3} R_1}{\ell_2} \right)^{-1} \left(\frac{4\pi a}{\ell_2^3} \right) (1 - C'\rho^\epsilon) N(\kappa^2 - 1) \geq 4\pi a \rho N(1 - C_0 \rho^\epsilon), \tag{4.49}$$

which was to be proved. □

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